Upsilon Invariant, Fibered Knots and Right-veering Open Books

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Ozsváth, Stipsicz and Szabó define a one-parameter family $\{\Upsilon_K(t)\}_{t\in[0,2]}$ of Heegaard Floer knot invariants for knots $K \subset S^3$. We generalize $\Upsilon_K(t)$ to knots in any rational homology sphere. We study the $\Upsilon-$invariant of a fibered knot. We prove that the $\Upsilon-$invariant can never reach its minimum slope if the monodromy of the fibration is not right-veering.

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To my parents

Chapter 1

Introduction

In [OSS17], Ozsváth, Stipsicz and Szabó define a one-parameter family $\{\Upsilon_K(t)\}_{t\in[0,2]}$ of Heegaard Floer knot invariants. $\Upsilon_K(t)$ is a knot concordance invariant. It bounds the 4-ball genus:

$$|\Upsilon_K(t)| \leq g_4(K)t.$$

Furthermore, they apply $\Upsilon_K(t)$ to the smooth concordance group $\mathcal{C}$. As an example, they show that the torus knot $T_{3,4}$ is linearly independent to any alternating knot in $\mathcal{C}$. In [OSS15], the authors prove that $\Upsilon_K(1)$ gives a lower bound for the smooth 4-dimensional crosscap number of K.

In this thesis, we generalize the $\Upsilon-$invariant to knots in rational homology spheres. For each $Spin^c-$structure $\mathfrak{s}$, we define the invariant $\Upsilon_{K,\mathfrak{s}}(t)$. Then we focus on the special case when K is a fibered knot.

In a similar setting, Grigsby, Licata and Wehrli [GLW16] define a family of annular Rasmussen invariants $\{d_t(L,o)\}_{t\in[0,2]}$ from the Khovanov-Lee complex of an oriented link in a thickened annulus. In particular, the authors study the case when (L,o) is a braid closure $\hat{\beta}$ equipped with its braid-like orientation. They find a rather interesting connection between $d_t(\hat{\beta})$ and the positivity of braids:

Theorem 1.0.1. *[GLW16] Let $\hat{\beta}$ be a braid closure with its natural orientation. If β is quasipositive, then $d'_t(\hat{\beta}) = b$ for all $t \in [0, 1)$, where b is the braid index of β.*

Theorem 1.0.2. *[GLW16] If $d'_t(\hat{\beta}) = b$ for some $t \in [0, 1)$, then β is right-veering.*

Inspired by the above theorems, the slope of the $\Upsilon-$invariant for fibered knots is of particular interest. Let Y be a rational homology sphere and let $K \subset Y$ be a fibered knot. The fibered surface Σ and the monodromy $\phi : \Sigma \to \Sigma$ define an open book decomposition (Σ, ϕ) on Y. By Giroux correspondence [Gir02], there is a one-to-one correspondence between open book decomposition up to positive stabilization and isotopy classes of contact structures ξ on Y. ξ induces a *Spin*^{c} structure $\mathfrak{s} = \mathfrak{s}(\xi)$ on Y.

Since the work of Honda, Katez and Matić [HKM07], the notion of right-veering (definition 3.1.1) of the monodromy ϕ plays a vital role in contact geometry due to the following theorem [HKM07]:

Theorem 1.0.3. *If ξ is tight, then every open book (Σ, ϕ) compatible with ξ is right-veering.*

Ozsváth and Szabó define the contact invariant in Heegaard Floer homology in [OS05]. The invariant is a class $c(\xi) \in \widehat{HF}(-Y, \mathfrak{s}(\xi))$ assigned to a contact structure ξ on Y. If $c(\xi) \neq 0$, then ξ is tight. It follows from theorem 1.0.3 that any open book compatible with ξ is right-veering. $c(\xi)$ does not detect right-veeringness completely, however; Honda, Katez and Matić prove that any contact structure admits a right-veering open book via positive stabilization [HKM07]. Moreover, Lisca [Lis11] shows that it is possible to have an overtwisted contact structure compatible with a right-veering open book which can not be destabilized. The following theorem attempts to further extract information from the knot Floer complex of the binding K by studying $\Upsilon_{K,\mathfrak{s}(\xi)}(t)$.

Theorem 1.0.4. *If $\Upsilon'_{K,\mathfrak{s}}(t) = -g$ for some $t \in [0,1)$, where g is the genus of the fibered surface Σ, then $\phi : \Sigma \to \Sigma$ is right-veering. The converse does not hold in general.*

This theorem is similar to theorem 1.0.2. However, the analogue of theorem 1.0.1 does not hold, as the $\Upsilon-$invariant doesn't necessarily have a single slope on $t \in [0,1)$ when ϕ is a product of positive Dehn twist. Indeed, let K be the torus knot $T(3,7)$, then $\Upsilon_K(t) = -6t$ for $t \in [0, \frac{2}{3}]$ and -4 for $t \in [\frac{2}{3}, 1)$.

Remark. A result of Hedden [Hed05] tells us that given a fibered knot $K \subset S^3$, the following are equivalent:

1. K is strongly quasi-positive;
2. $\tau(K) = g(K)$;
3. the fibration is compatible with the unique tight contact structure on S^3.

1 and 2 combined with the fact that $\Upsilon_K(t) = -\tau(K)t$ [OSS17] at $t = 0$ show that $\Upsilon_K(t) = -gt$ at $t = 0$, so the monodromy is right-veering, which also follows from 3. Unfortunately, we are unable to find any example such that $\Upsilon'_K(t) \neq -g(K)$ at 0 and $\Upsilon'_K(t) = -g(K)$ for some $t \in (0,1)$. Such an example will provide a fibered knot with right-veering monodromy but supports overtwisted contact structure.

1.1 Structure of this thesis

The remainder of this thesis is organized as follows. In chapter 2 we first briefly review the construction of the Knot Floer complex. We focus on definitions and constructions that are necessary for our purpose. Then we define the generalized $\Upsilon-$invariant and establish some basic properties of Υ. In chapter 3 we review the definition of right-veeringness and study the case for fibered knots. Then we prove theorem 1.0.4 and provide some examples.

Chapter 2

Generalized $\Upsilon-$ Invariant

2.1 Knot Floer complex

In this section we briefly review the construction of the Heegaard Floer complex of knots following [OS04] and [Ras03]. Let Y be a rational homology sphere, and let $K \subset Y$ a null-homologous knot. We can associate to the pair (Y, K) a 2-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ consisting of the following data:

- A Heegaard surface of genus g, splitting Y into two handlebodies U_0 and U_1;
- linearly independent curves $\alpha = \{\alpha_1, ..., \alpha_g\}$, $\beta = \{\beta_1, ..., \beta_g\}$ on Σ;
- Based points $w, z \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g$.

Connect w and z by a curve a in $\Sigma - \alpha_1 - ... - \alpha_g$ and another curve in $\Sigma - \beta_1 - ... - \beta_g$. The knot K is obtained by pushing a and b into U_0 and U_1 respectively. One can always construct such a 2-pointed diagram from a suitable Morse function on the knot complement.

Let $\Sigma^{\times g}$ be the Cartesian product of g copies of Σ. The symmetric product $Sym^g(\Sigma)$ is obtained from $\Sigma^{\times g}$ quotient by the symmetric group S_g, which acts on $\Sigma^{\times g}$ by permutation. In other words $Sym^g(\Sigma)$ consists of unordered g-tuples of points in Σ. Inside $Sym^g(\Sigma)$ there are two half-dimensional tori:

$$\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g / S_g, \ \mathbb{T}_\beta = \beta_1 \times ... \times \beta_g / S_g$$

.

A complex structure on Σ induces one on $Sym^g(\Sigma)$, where $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are totally real. Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be two intersection points, and let $\pi_2(\mathbf{x}, \mathbf{y})$ be the set of relative homotopy classes of disks

$$u : D^2 \to Sym^g(\Sigma),$$

with $u(-1) = \mathbf{x}$, $u(1) = \mathbf{y}$, and the lower half of ∂D^2 mapping to $\mathbb{T}_\alpha$ and the upper half to $\mathbb{T}_\beta$. For each $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, let $\mathcal{M}(\phi)$ be the moduli space of J-holomorphic representatives of ϕ, where J is an almost complex structure on $Sym^g(\Sigma)$. $\mathcal{M}(\phi)$ admits an $\mathbb{R}-$action, and we denote the quotient space by $\widehat{\mathcal{M}}(\phi)$. The dimension of $\widehat{\mathcal{M}}(\phi)$ is called the Maslov index $\mu(\phi)$.

Let $C(K)$ be the free abelian group generated by intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. $C(K)$ has two gradings: the Maslov (homological) grading and Alexander grading. Let $n_w(\phi) = \#\phi^{-1}(\{w\} \times Sym^{g-1}(\Sigma))$ and $n_z(\phi) = \#\phi^{-1}(\{z\} \times Sym^{g-1}(\Sigma))$. $n_w(\phi), n_z(\phi)$ are well-defined since $\{w\} \times Sym^{g-1}(\Sigma)$ and $\{z\} \times Sym^{g-1}(\Sigma)$ are both disjoint from $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. The Alexander grading $A(\mathbf{x})$ is characterized by:

- the function $A(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi)$;

- the Euler characteristic $\triangle_K(T) = \sum_a \sum_m (-1)^m rank(H_{a,m}(K))T^m = \triangle_K(T^{-1})$, where a is the Alexander grading and m is the Maslov grading.

Now we can define the knot Floer complex $CFK^\infty(Y, K)$:

- over $\mathbb{F}_2[U, U^{-1}]$,
- whose generators are elements of the form $[\mathbf{x}, i, j]$, where $j - i$ is the Alexander grading of $\mathbf{x}$,
- whose differential is given by

$$\partial^\infty[\mathbf{x}, i, j] = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y}|\mu(\phi)=1\}} \#(\widehat{M}(\phi))[\mathbf{y}, i - n_w(\phi), j - n_z(\phi)]$$

  where $\#(\widehat{\mathcal{M}}(\phi))$ is counted modulo 2,
- with U-action $U([\mathbf{x}, i, j]) = [\mathbf{x}, i - 1, j - 1]$,
- splitting as a direct sum:

$$CFK^\infty(Y, K) = \bigoplus_{\mathfrak{s}\in spin^c(Y)} CFK^\infty(Y, K, \mathfrak{s})$$

  where $\mathfrak{s}$ runs over $Spin^c$ structures on Y.

The homology of $CFK^\infty(Y, K, \mathfrak{s})$ is $HF^\infty(Y, \mathfrak{s}) \cong \mathbb{F}[U, U^{-1}]$ as a relatively graded $\mathbb{F}[U, U^{-1}]-$module. An absolute grading can be defined where the base element $\mathbf{1} \in \mathbb{F}[U, U^{-1}]$ has Maslov (homological) grading $d(Y, \mathfrak{s})$, which is the Heegaard Floer correction term [OS03]. The $U-$action changes the Maslov grading by -2. There is a $\mathbb{Z} \oplus \mathbb{Z}$ filtration on $CFK^\infty(Y, K, \mathfrak{s}) = C$ given by the map $[\mathbf{x}, i, j] \mapsto [i, j]$, where (i, j) corresponds to the algebraic and Alexander filtration respectively. $i = 0$ is the minimum algebraic filtration level such that the image of the inclusion induced map on homology $H(C\{i \leq k\}) \hookrightarrow H(C)$ contains the base element of degree $d(Y, \mathfrak{s})$.

Remark. It follows from [OS04] and [Ras03] that $CFK^{\infty}(Y, K)$ is independent of the choices of the 2-pointed Heegaard diagram and generic almost complex structure J in the sense that different choices yield chain homotopy equivalent. From a different perspective, if one equips $Sym^g(\Sigma)$ with a symplectic form, then the above construction defines the Lagrangian Floer homology of the pair $(\mathbb{T}_\alpha, \mathbb{T}_\beta)$, whose differential counts J-holomorphic disks in $Sym^g(\Sigma)$. Gromov started the theory of J-holomorphic curve [Gro85]. The construction of Floer homology was first provided by Floer [Flo88].

2.2 The Definition and Properties of the $\Upsilon-$invariant

In this section we generalize the definition of the $\Upsilon-$invariant for K a null-homologous knot in a rational homology sphere based on Livingston's approach in [Liv17]. We also develop necessary machinery for later discussion related to open book decomposition and contact structure.

2.2.1 t-filtration and Υ

Fix $t \in [0, 2]$ and a generator $[\mathbf{x}, i, j]$, we start with a real-valued function

$$f_t([\mathbf{x}, i, j]) = (1 - \frac{t}{2})i + \frac{t}{2}j$$

on $CFK^{\infty}(Y, K, \mathfrak{s}) = C$. Furthermore, let $\theta = [\mathbf{x}_1, i_1, j_1] + ... + [\mathbf{x}_n, i_n, j_n]$ be a chain in C, we also define a function

$$F_t(\theta) = max\{f_t([\mathbf{x}_k, i_k, j_k)]\}.$$

Proposition 2.2.1. *F_t defines a filtration $\mathcal{F}^t$ on C, where the fitered subcomplexes are given by $\mathcal{F}^t_s = f_t^{-1}(-\infty, s]$. Furthermore, $\mathcal{F}^t$ is discrete, i.e., for any $s_1 \geq s_2$, $\mathcal{F}^t_{s_1}/\mathcal{F}^t_{s_2}$ is finite-dimensional.*

Proof. Under the boundary map $\partial^\infty(\theta) = \Sigma\partial^\infty[\mathbf{x}_k, i_k, j_k]$, where ∂^∞ reduce both i_k and j_k. Both $1-\frac{t}{2}$ and $\frac{t}{2}$ are positive as well so that $F_t(\theta) \geq F_t(\partial^\infty(\theta))$.

For discreteness we see that there are k_1 and k_2 such that $C(i \leq k_1) \subset \mathcal{F}^t_{s_2} \subset \mathcal{F}^t_{s_1} \subset C(i \leq k_2)$. Since the algebraic filtration is discrete, so is $\mathcal{F}^t$ ■

Definition 2.2.2. $\nu_t(Y, K, \mathfrak{s}) = min$ *$\{F_t(\theta)|\theta$ is a cycle in C and $[\theta]$ is non-trivial with Maslov grading $d(Y, \mathfrak{s})\}$.*

We can see that $\nu_t(Y, K, \mathfrak{s})$ is in fact the minimum $\mathcal{F}^t-$filtered level such that the inclusion induced map $H(\mathcal{F}_t) \hookrightarrow H(C)$ on homology contains the base element with degree $d(Y, \mathfrak{s})$.

Definition 2.2.3. $\Upsilon_{Y,K,\mathfrak{s}}(t) = -2\nu_t(Y, K, \mathfrak{s})$.

When Y is understood from the context, then we drop it from the notation. We say a generator $[\mathbf{x}, i, j]$ realizes $\Upsilon_{K,\mathfrak{s}}(t)$ if $[\mathbf{x}, i, j]$ is a summand of a cycle θ satisfying the condition in definition 2.2.2 and $\nu_t(K, \mathfrak{s}) = f_t([\mathbf{x}, i, j])$.

2.2.2 Υ as a function of t

An initial observation is that $\Upsilon_{K,\mathfrak{s}}(0) = 0$. Indeed, $f_0([\mathbf{x}, i, j]) = i$ is the algebraic filtration.

Theorem 2.2.4. *Given $t \in [0, 2]$,*

(a) $\Upsilon_{K,\mathfrak{s}}(t)$ is a continuous piece-wise linear function.

(b) If $\Upsilon_{K,\mathfrak{s}}(t)$ is differentiable at t, and a generator $[\mathbf{x}, i, j]$ realizes $\Upsilon_{K,\mathfrak{s}}(t)$, then $\Upsilon'_{K,\mathfrak{s}}(t) = i - j = -A(\mathbf{x})$.

(c) $\Upsilon_{K,\mathfrak{s}}(t)$ is not differentiable at t only if at least two generators $[\mathbf{x}, i, j]$, $[\mathbf{x}', i', j']$ realize $\Upsilon_{K,\mathfrak{s}}(t)$.

Proof. The proof is essentially the same as [Liv17]. Since $\mathcal{F}^t$ is discrete, for all but finitely many t there is exactly one generator $[\mathbf{x}, i, j]$ realizing $\Upsilon_K(t)$. For nearby t, say t_1, $\Upsilon_K(t_1)$ is realized by the same generator $[\mathbf{x}, i, j]$ so that $\nu_{t_1}(K, \mathfrak{s}) = (1 - \frac{t_1}{2})i + \frac{t_1}{2}j$. Written differently,

$$\Upsilon_{K,\mathfrak{s}}(t) = -2\nu_t(K, \mathfrak{s}) = (i - j)t - 2i.$$

Thus $\Upsilon'_{K,\mathfrak{s}}(t) = i - j$. Furthermore, $\Upsilon_{K,\mathfrak{s}}(t)$ is not differentiable only if two generators $[\mathbf{x}, i, j]$, $[\mathbf{x}', i', j']$ realize $\Upsilon_{K,\mathfrak{s}}(t)$ and $i - j \neq i' - j'$. ■

Corollary 2.2.5. *$\Upsilon'_{K,\mathfrak{s}}(t)$ is between $-g(k)$ and $g(k)$.*

Proof. The Alexander grading is always between $-g(K)$ and $g(K)$.

Theorem 2.2.6. *The $\Upsilon-$invariant satisfies the following properties:*

(a) $\Upsilon_{Y\#Y', K\#K', \mathfrak{s}\#\mathfrak{s}'}(t) = \Upsilon_{Y,K,\mathfrak{s}}(t) + \Upsilon_{Y',K',\mathfrak{s}'}(t)$.

(b) $\Upsilon_{Y,K,\mathfrak{s}}(t) = -\Upsilon_{-Y,K,\mathfrak{s}}(t)$

(c) $\Upsilon_{K,\mathfrak{s}}(t) = \Upsilon_{K,\mathfrak{s}}(2 - t)$.

Proof. For part (a), the complex $CFK^\infty(Y\#Y', K\#K', \mathfrak{s}\#\mathfrak{s}')$ is bifiltered chain homotopy equivalent to $CFK^\infty(Y, K, \mathfrak{s}) \otimes CFK^\infty(Y', K', \mathfrak{s}')$. If $(C, \mathcal{F})$ and $(C', \mathcal{F}')$ are two filtered complexes, there is a natural filtration $\mathcal{F} \otimes \mathcal{F}'$ on $C \otimes C'$:

$$(C \otimes C')_s = Image(\oplus_{s=s_1+s_2} C_{s_1} \otimes C'_{s_2} \to C \otimes C').$$

It follows from theorem 6.1 in [Liv17] that ν_t is additive for each t. Hence $\Upsilon_{Y\#Y',K\#K',\mathfrak{s}\#\mathfrak{s}'}(t) = \Upsilon_{Y,K,\mathfrak{s}}(t) + \Upsilon_{Y',K',\mathfrak{s}'}(t)$.

For part (b), the complex $CFK^\infty(Y,K,\mathfrak{s})$ with filtration $\mathcal{F}^t$ has a dual complex $CFK^\infty(Y,K,\mathfrak{s})^*$ with decreasing filtration $\mathcal{F}^{t*}$. $\nu_t(K)$ can be defined as the maximal filtration level of a class in the dual complex which contains a non-trivial element of cohomology in grading $d(Y,\mathfrak{s})$. Since $(CFK^\infty(-Y,K,\mathfrak{s}),\mathcal{F}) \cong (CFK^\infty(Y,K,\mathfrak{s})^*, -\mathcal{F}^*)$. $\Upsilon_{Y,K,\mathfrak{s}}(t) = -\Upsilon_{-Y,K,\mathfrak{s}}(t)$ is proved.

Part (c) follows immediately from switching the role of base points w and z. ■

Chapter 3

The $\Upsilon-$invariant of Fibered Knots

In this chapter we prove Theorem 1.0.4.

Theorem 3.0.1. *If $\Upsilon'_{K,\mathfrak{s}}(t) = -g$ for some $t \in [0,1)$, where g is the genus of the fibered surface Σ, then $\phi : \Sigma \to \Sigma$ is right-veering. The converse does not hold in general.*

We start this chapter by reviewing the definition of right-veering surface diffeomorphism [HKM07].

3.1 Right-veering diffeomorphism

Let Σ be a compact oriented surface with boundary $\partial\Sigma$, and let α, $\beta : [0,1] \to \Sigma$ be properly embedded oriented arcs with $\alpha(0) = \beta(0) = x \in \partial\Sigma$. Isotope α and β so that they intersect transversely with the fewest possible number of intersections. We say that β is to the right of α if $(\dot{\beta}(0), \dot{\alpha}(0))$ define the orientation of Σ at x.

Definition 3.1.1. *Let $\phi : \Sigma \to \Sigma$ be a diffeomorphism which restricts to the identity map on the boundary $\partial\Sigma$. Let α be a properly embedded oriented arc starting at a* *based point $x \in \partial\Sigma$. Then we say ϕ is right-veering if for arbitrary based point x and arc α, $\phi(\alpha)$ is always to the right of α.*

3.2 Knot Floer homology of fibered knots

Let K be the binding of an open book (Σ, ϕ) of Y compatible with a contact structure ξ. A basis for Σ is a collection $\{a_1, ..., a_{2g}\}$ of disjoint, properly embedded arcs in Σ whose complement is a disk. Let b_i be an isotopic copy of a_i obtained by shifting the end points of a_i in the direction of K so that b_i intersects a_i at a single point x_i. Following [HKM09], we form a pointed Heegaard diagram

$$(S, \beta = (\beta_1, ..., \beta_{2g}), \alpha = (\alpha_1, ..., \alpha_{2g}), w)$$

for $-Y$ by doubling the open book:

- $S = \Sigma \cup -\Sigma$ is the union of two copies of Σ glued along the binding K,
- $\alpha_i = a_i \cup a_i$,
- $\beta_i = b_i \cup \phi(b_i)$,
- the based point w lies outside of the strip from the isotopies from a_i to b_i

as shown in the following figure,

Now we turn the Heegaard diagram into a doubly-pointed Heegaard diagram for $K \subset -Y$. We perform finger moves on the β curves in the direction of the orientation of K, and place the second based point z inside the region of the isotopies.

The following lemma by Baldwin and Vela-Vick [BVV18] characterize the Alexander grading of generators.

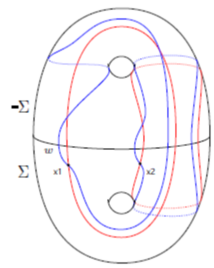

Figure 3.1: the arcs a_1, a_2 are red and b_1, b_2 are blue. The intersection points x_1, x_2 are shown in black dots.

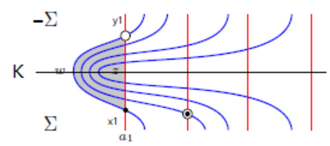

Figure 3.2: A doubly-pointed Heegaard diagram of $K \subset -Y$. The bigon from $\mathbf{y}$ to $\mathbf{x}$ is shown in grey.

Lemma 3.2.1. *The Alexander grading of a generator* $\mathbf{x}$ *is the number of components in* $-\Sigma \subset S$ *minus* g.

Proposition 3.2.2. *If* $A(\mathbf{x}) = -g$, *then every component* $\mathbf{x}$ *lies in* Σ, *which is an intersection provided by the finger moves.*

If ϕ is not right-veering, then from [HKM09] there exists a non-separating arc a_1 such that $\phi(a_1)$ is to the left of a_1. a_1 can be completed to a basis $\{a_1, ...a_{2g}\}$.

Corollary 3.2.3. *Given a generator* $\mathbf{x}$ *with* $A(\mathbf{x}) = -g$, *if* ϕ *is not right-veering, then there is a bigon containing the based point* z *that connects some other generator* $\mathbf{y}$ *to* $\mathbf{x}$. *Moreover,* $A(\mathbf{y}) = 1 - g$.

Proof. See Figure 3.2. Notice that on $-\Sigma$, $\phi(b_1)$ is to the right of b_1. ■

3.3 Proof of Theorem 1.0.4

We will prove the following: if $\phi : \Sigma \to \Sigma$ is not right-veering and $\Upsilon'_{K,\mathfrak{s}}(t) = -g$ then $t \geq 1$. In fact, we will show that if $\Upsilon'_{m(K),\mathfrak{s}}(t) = g$ then $t \geq 1$, where $m(K)$ is the mirror of K. Then the theorem follows from theorem 2.2.6 that $\Upsilon_{K,\mathfrak{s}}(t) = -\Upsilon_{m(K),\mathfrak{s}}(t)$.

Now we consider the complex $CFK^\infty(-Y, K, \mathfrak{s})$ associated to the Heegard diagram compatible with the open book (Σ, ϕ).

Suppose $\Upsilon'_{m(K),\mathfrak{s}}(t_0) = g$ for some t_0. It follows from Theorem 2.2.4 that $U^m\mathbf{c}$ realizes $\nu_{t_0}(-Y, K, \mathfrak{s})$, where $\mathbf{c}$ is a chain with $A(\mathbf{c}) = -g$. We recall the definition:

Definition 3.3.1. $\nu_t(-Y, K, \mathfrak{s}) = min$ $\{F_t(\theta)|\theta$ *is a cycle in* C *and* $[\theta]$ *has Maslov grading* $d(-Y, \mathfrak{s})\}$.

If $\theta = \sum[\mathbf{x}_k, i_k, j_k]$, then $F_t(\theta) = max\{f_t([\mathbf{x}_k, i_k, j_k)]\}$.

There exists some cycle $\eta \in CFK^\infty(-Y, K, \mathfrak{s})$ satisfying:

- $[\eta] \in HFK^\infty(-Y, K, \mathfrak{s})$ has absolute grading $d(-Y, \mathfrak{s})$.
- $\eta = U^m\mathbf{c} + \eta'$
- $\nu_{t_0}(-Y, K, \mathfrak{s}) = F_{t_0}(\eta') = f_{t_0}(U^m\mathbf{c}) = m - \frac{gt_0}{2} \geq F_{t_0}(\eta')$.

Suppose $F_{t_0}(\eta') = (1 - \frac{t_0}{2})i + \frac{t_0}{2}j$ for some (i, j). Hence,

$$m - \frac{gt_0}{2} \geq i - \frac{i-j}{2}t_0.$$

Since $i - j \leq g$, the inequality holds for any $0 \leq t_1 < t_0$. Thus,

$$F_{t_1}(\eta) = f_{t_1}(U^m\mathbf{c}) = m - \frac{gt_1}{2}$$

for any $0 \leq t_1 < t_0$. In other words, any summand of η other than $U^m\mathbf{c}$ can only realize the $\Upsilon-$invariant when $t > t_0$.

If $\phi : \Sigma \to \Sigma$ is not right-veering, then from proposition 3.2.3 there is a generator $\mathbf{y}$ such that

- $\partial^\infty(U^m\mathbf{y}) = U^m\mathbf{c} + \theta$, and
- $\mathrm{A}(\mathbf{y}) = 1 - g$.

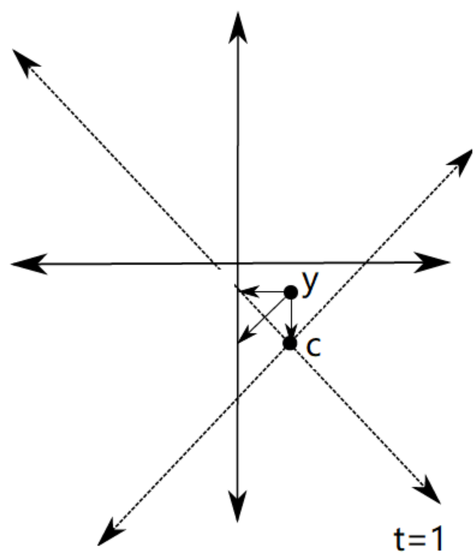

Figure 3.3: This figure shows that we have other generators realizing the $\Upsilon-$invariant for some $1 \leq t \leq t_0$ if there is a bigon from $\mathbf{y}$ to $\mathbf{c}$

.

Then

$$\partial^\infty(\partial^\infty U^m\mathbf{y}) = \partial^\infty U^m\mathbf{c} + \partial^\infty\theta = 0.$$

Since η is a cycle,

$$\partial^\infty\eta = \partial^\infty U^m\mathbf{c} + \partial^\infty\eta' = 0$$

as well. Therefore, $\theta+\eta'$ is also a cycle in $CFK^\infty(-Y,K,\mathfrak{s})$, denoted by δ. Moreover, δ has Maslov grading $d(-Y,\mathfrak{s})$ and

$$F_{t_0}(\delta) = max(F_{t_0}(\theta), F_{t_0}(\eta')) \geq F_{t_0}(\eta) = f_{t_0}(U^m\mathbf{c})$$

because $F_{t_0}(\eta) = \nu_t(-Y,K,\mathfrak{s}) = \min\,\{F_t(\theta)|\theta$ is a cycle in C and $[\theta]$ has Maslov grading $d(-Y,\mathfrak{s})\}$. Thus, $F_{t_0}(\theta) \geq f_{t_0}(U^m\mathbf{c}) \geq F_{t_0}(\eta')$. Suppose $F_{t_0}(\theta) = (1-\frac{t_0}{2})i' + \frac{t_0}{2}j'$ for some (i',j'). Hence,

$$m - \frac{gt_0}{2} \leq i' - \frac{i'-j'}{2}t_0.$$

Again $i'-j' \leq g$. There is a bigon containing z from $\mathbf{y}$ to $\mathbf{c}$, so $\mathbf{y}$ and $\mathbf{c}$ are at the same algebraic filtered level. Since $\partial^\infty(U^m\mathbf{y}) = U^m\mathbf{c}+\theta$ and ∂^∞ reduce algebraic filtered level, we conclude that $m \geq i'$. Therefore, there exists $t_2 < t_0$,

$$m - \frac{gt_2}{2} = i' - \frac{i'-j'}{2}t_2.$$

Rewrite it as

$$t_2 = \frac{2(m-i')}{g-(i'-j')}.$$

Moreover, for some $t' \in (t_2-\epsilon, t_2)$, $\nu_{t'}(-Y,K,\mathfrak{s}) = F_{t'}(\theta)$ is realized by some generator $[\mathbf{x},i',j']$. Since $\partial^\infty(U^m\mathbf{y}) = U^m\mathbf{c}+\theta$ and $\mathrm{A}(\mathbf{y}) = 1-g$,

$$A(\mathbf{x}) = j'-i' \geq 2-g$$

and

$$m-i' \geq j'-i'-(1-g).$$

Therefore,

$$t_2 \geq \frac{2(j'-i'-(1-g))}{g-(i'-j')} = 2 - \frac{2}{g-(i'-j')} \geq 1.$$

and $t_0 \geq t_2 \geq 1$ as desired. ■

3.4 Examples

Example 3.4.1. *For fibered knots $K \subset S^3$ with less than 10 crossings, $\Upsilon'_K(t) = -g$ for some $t \in [0, 1)$ if and only if K supports the unique tight contact structure on S^3.*

Proof. For any knot in S^3, Ozsváth and Szabó [OSS17] prove that $\Upsilon_K(t) = -\tau(K)t$ for small t. Moreover, if the fibered knot supports the unique tight contact structure on S^3, then $\tau(K) = g(K)$.

On the other hand, a fibered knot K supports a tight contact structure in S^3 if and only if it is strongly quasi-positive. We look up the monodromies of fibered knots under 10 crossings that are not strongly quasi-positive from [Kno]. By brute force we find that none of them are is right-veering. Therefore, by theorem 1.0.4, $\Upsilon'_K(t) > -g$. ■

Example 3.4.2. *The converse of theorem 1.0.4 does not hold even for fibered knots in S^3.*

Proof. Let us consider the knot $K = 8_{20}$, which is a slice and fibered knot. The $(p, 1)-$cable $K_{p,1}$ is also slice and fibered. Indeed, 8_{20} is the pretzel $P(3, -3, 2)$. One can construct a slice disks by adding two 1-handles and three 2-handles in B^4. The slice disk of the $(p, 1)-$cable can be obtained by stacking p copies of the disks constructed above and connecting them with half-twisted bands. Therefore, $\Upsilon_{K_{p,1}}(t) = 0$. On the other hand, Kazez and Roberts [KR12] show that the fractional Dehn twist coefficient of a fibered knot obtained by cabling is $\frac{1}{p} > 0$. Hence the monodromy of K is right-veering.

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